1. Warmup: Permanent Income Hypothesis

Solve the deterministic individual problem

\[
\max_{\{c_t, a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)
\]

\[\text{s.t. } c_t + a_{t+1} = y_t + (1 + r)a_t \quad \forall t\]

f.o.n.c.'s are

\[c_t : \quad u'(c_t) = \lambda_t\]

\[a_{t+1} : \quad \lambda_t = \beta(1 + r)\lambda_{t+1}\]

Assuming \(\beta(1 + r) = 1\), we obtain

\[u'(c_t) = u'(c_{t+1})\]

So if \(u(\cdot)\) is strictly concave, \(c_{t+j} = c_t\) for all \(j\). Then writing out the budget constraints,

\[c_t + a_{t+1} = y_t + (1 + r)a_t\]
\[c_t + a_{t+2} = y_{t+1} + (1 + r)a_{t+1}\]
\[c_t + a_{t+3} = y_{t+2} + (1 + r)a_{t+2}\]
\[c_t + a_{t+4} = y_{t+3} + (1 + r)a_{t+3}\]

\[\ldots\]

Multiply iteratively by \(\frac{1}{1+r}\) to get

\[c_t + a_{t+1} = y_t + (1 + r)a_t\]

\[\frac{1}{1+r} c_t + \frac{1}{1+r} a_{t+2} = \frac{1}{1+r} y_{t+1} + a_{t+1}\]
\[
\left( \frac{1}{1+r} \right)^2 c_t + \left( \frac{1}{1+r} \right)^2 a_{t+3} = \left( \frac{1}{1+r} \right)^2 y_{t+2} + \left( \frac{1}{1+r} \right) a_{t+2}
\]
\[
\left( \frac{1}{1+r} \right)^3 c_t + \left( \frac{1}{1+r} \right)^3 a_{t+4} = \left( \frac{1}{1+r} \right)^3 y_{t+3} + \left( \frac{1}{1+r} \right)^2 a_{t+3}
\]
\[\ldots\]

Then adding up LHS’s and RHS’s, all the a’s cancel out, so

\[
\sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j c_t + \lim_{J \to \infty} \left( \frac{1}{1+r} \right)^j a_{t+j+1} = \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j y_{t+j} + (1+r)a_t
\]

By the TVC or borrowing constraint, the last term of LHS is 0. Hence

\[
\frac{1+r}{r} c_t = \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j y_{t+j} + (1+r)a_t
\]

So

\[
c_t = \frac{r}{1+r} \cdot \left\{ y_t + \sum_{j=1}^{\infty} \left( \frac{1}{1+r} \right)^j y_{t+j} + (1+r)a_t \right\}.
\]

Friedman’s conjecture is that this holds even in the stochastic case:

\[
c_t = \frac{r}{1+r} \cdot \left\{ y_t + \mathbb{E}_t \sum_{j=1}^{\infty} \left( \frac{1}{1+r} \right)^j y_{t+j} + (1+r)a_t \right\}.
\]

This is called "certainty equivalence," a principle that is exploited in many other applications as well to show that the solution to a stochastic problem coincides with its deterministic counterpart. Of course, it is something that has to be shown case by case, not a universal property.

2. The Income Fluctuation Problem

This is a summary of some important propositions from Huggett (1993); Aiyagari (1993); Chamberlain and Wilson (2000).
2.1 $\beta(1 + r) < 1$ w/o uncertainty

Assume a deterministic income process $\{y_t\}_{t=0}^{\infty}$ s.t. $y_t = \bar{y}$ for all $t$, i.e., the endowment is same every period. The individual’s problem is

$$\max_{\{c_t, a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

s.t. $c_t + a_{t+1} = \bar{y} + (1+r)a_t$

$a_{t+1} \geq -B \quad \forall t$,

$$\lim_{J \to \infty} \left( \frac{1}{1+r} \right)^J a_{t+J+1} = 0 \quad \text{(TVC)}.$$

where $B$ is an (exogenous) borrowing limit. When the borrowing constraint is not binding, we get the usual Euler Equation

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) < u'(c_{t+1})$$

$$\Rightarrow c_t > c_{t+1}$$

as long as $u$ is concave. So consumption decreases over time. This implies that savings decrease over time as well, because

$$c_{t+1} = \bar{y} + (1+r)a_{t+1} - a_{t+2}, \quad c_t = \bar{y} + (1+r)a_t - a_{t+1}$$

$$\Rightarrow 0 > c_{t+1} - c_t = (1+r)(a_{t+1} - a_t) - (a_{t+2} - a_{t+1})$$

$$\Rightarrow a_{t+2} - a_{t+1} > (1+r)(a_{t+1} - a_t)$$

$$\Rightarrow \left( \frac{1}{1+r} \right) (a_{t+2} - a_{t+1}) > a_{t+1} - a_t.$$

Then iterating forward,

$$0 = \lim_{J \to \infty} \left( \frac{1}{1+r} \right)^J (a_{t+J+1} - a_{t+J}) > a_{t+1} - a_t$$

$$a_t > a_{t+1}.$$
So both consumption and savings decrease; consumption cannot be negative and debt cannot go over the borrowing limit. When the borrowing constraint is binding, then,

\[ a_{t+1} = -B \]
\[ c_t = \bar{y} - rB. \]

So once you hit the borrowing limit, you cannot borrow anymore, continue to pay the interest to your debt, and consume whatever is left over. So we can define a natural borrowing limit \( B = \frac{\bar{y}}{r} \) which you would never hit (as long as zero consumption is not allowed)—consumption just decreases forever toward zero!

2.2 \( \beta(1 + r) = 1 \) w/ uncertainty: Prudence and Precautionary Savings

Now \( \{y_t\} \) follows a stochastic process with constant mean (i.e., on average the endowment is the same). The Euler equation for the individual is

\[ u'(c_t) = \beta (1 + r) \mathbb{E}_t u'(c_{t+1}) \geq u'(\mathbb{E}_t c_{t+1}), \]

If the inequality holds with >, then \( c_t < \mathbb{E}_t c_{t+1} \), so consumption is a submartingale while MU is a supermartingale. By the martingale convergence thm, both \( c_t \) and \( u'(c_t) \) must converge, the question is where. Turns out that \( u'(c_t) \to 0 \) (why?), \( c_t \to \infty \), and \( a_t \to \infty \). (Think about it: If I want to continue to increase consumption, I have to save more every period. We could derived the same using recursive methods, which we’re still abstracting away from for now...)

When do we have \( \mathbb{E}_t u'(c_{t+1}) > u'(\mathbb{E}_t c_{t+1})? \) This holds when \( u' \) is convex. Hence, \( u \) being concave is not enough, or put differently, high risk aversion is not enough even though it may seem like it would. For example, quadratic utility will not achieve this: Suppose \( u(c) = -c^2 \), then \( u'(c) = -2c \), so

\[ -2c_t = \mathbb{E}_t [-2c_{t+1}] \]
\[ c_t = \mathbb{E}c_{t+1}. \]
and we would get certainty equivalence—i.e., Friedman’s PIH in (1) becomes true, not just a conjecture. What we need is \( u' \) being convex, or, \( u''' > 0 \), which we call prudence. If the utility function displays prudence, the savings behavior displays precautionary savings, so that I save more than I would without uncertainty (certainty equivalence does not hold). In fact, this is how Friedman’s PIH was disproved.

### 2.3 \( \beta(1 + r) < 1 \) w/ uncertainty

This is the most important case. Suppose that the endowment process is Markov, i.e., that \( y' \sim F(y'|y) \). To make things simpler, suppose that \( y \) can only take on a \( n \) number of values, so that the Markov process can be expressed as a transition matrix \( \Pi_{n \times n} \): Each element tells you the probability that the next period endowment is \( y_j \) if the current period endowment is \( y_i \). The savings problem, now in recursive form, is

\[
v(a, y_i) = \max_{a' \geq -B} \left\{ u(c) + \beta \mathbb{E}_{y_i} v(a', y_j) \right\} \quad \text{s.t.} \quad c + a' \leq y_i + (1 + r)a
\]

\[
= \max_{a' \geq -B} \left\{ u((1 + r)a + y_i - a') + \beta \sum_{j=1}^{n} \pi_{i,j} v(a', y_j) \right\}
\]

where I have just replaced \( c \) in the period utility function and explicitly expressed the expectation. Now, let us redefine some variables as follows. Let

\[
A = a + B
\]

\[
Z = (1 + r)a + B + y_i = (1 + r)A + y_i - rB,
\]

so then we can write

\[
V(Z, y_i) = \max_{A' \geq 0} \left\{ u(Z - A') + \beta \sum_{j} \pi_{i,j} V((1 + r)A' + y_j - rB, y_j) \right\}.
\]  

(2)

Note that we only need to keep \( y_i \) in today’s state and \( y_j \) in tomorrow’s state because it is needed to know which elements of \( \pi_{i,j} \) we should sum over in the expectation. Other than that, it no longer contains any relevant information about how much wealth I have today and how much I can borrow/save; it is all compressed into \( (A, Z) \). Here, \( Z \) is “cash-at-hand” or liquidity: It is the total amount I can eat taking into consideration how much I can borrow. \( A \)
is “net investment”: How much I can set aside for tomorrow.

We can now prove an important result:

**Theorem 1** Assets are bounded above if absolute risk aversion converges to 0, i.e.

\[ \lim_{c \to \infty} \frac{u''(c)}{u'(c)} = 0. \]

**Proof:** We can rewrite (2) as

\[
V(Z) = \max_{c,A' \in [0,Z]} \{ u(c) + \beta \mathbb{E} V[(1 + r)A' + y' - rB] \} \quad \text{s.t.} \quad c + A' \leq Z.
\]

where \( c(Z), A'(Z) \) are the optimal allocations for the second maximization problem in (2). Assume \( u \) and \( V \) are strictly increasing, concave and differentiable. Then (2) can be viewed as a standard 2-good utility maximization problem where \( Z \) is your budget, and \( c \) and \( A' \) are normal goods. Hence the optimal solutions \( c(Z) \) and \( A'(Z) \) are both (strictly) increasing in \( Z \).

Now we need

**Lemma 1** \( \exists Z^* \) s.t. for all \( Z \geq Z^* \), \( Z' \leq Z'_{\text{max}} = (1 + r)A'(Z) + y_{\text{max}} - rB \leq Z \), where \( y_{\text{max}} \) is the highest possible realization of income.

Intuitively, the meaning of the claim is this: Suppose my cash-at-hand exceeds \( Z^* \) today. Then even if I get the highest possible income shock, I will decrease my cash-at-hand tomorrow. Similarly we can define \( Z'_{\text{min}} = (1 + r)A'(Z) + y_{\text{min}} - rB \).

**Proof:** The Euler equation for \( V \) is

\[
u'(c(Z)) = \beta (1 + r) \mathbb{E} u'(c(Z'))
\]

\[
= \beta (1 + r) \frac{\mathbb{E} u'(c(Z'))}{u'(c(Z'_{\text{max}}))} u'(c(Z'_{\text{max}})) < \frac{\mathbb{E} u'(c(Z'))}{u'(c(Z'_{\text{max}}))} u'(c(Z'_{\text{max}})).
\]

\[1\text{See Appendix for proof.}\]
As \( Z \to \infty \), \( A'(Z) \to \infty \) since \( A'(Z) \) is increasing in \( Z \), so \( Z_{\text{max}}' \to \infty \). If we can show that 

\[
\frac{E u'(c(Z'))}{u'(c(Z'_{\text{max}}))} \to 1 \text{ as } Z'_{\text{max}} \to \infty
\]

we are done, since then

\[
\lim_{Z \to \infty} u'(c(Z)) \leq \lim_{Z \to \infty} u'(c(Z'_{\text{max}}))
\]

so that for some \( Z^* \) large enough,

\[
u'(c(Z)) \leq u'(c(Z'_{\text{max}})) \iff c(Z) \geq c(Z'_{\text{max}}) \iff Z \geq Z'_{\text{max}} \geq Z'
\]

for all \( Z \geq Z^* \), as desired. Now

\[
1 \leq \frac{E u'(c(Z'))}{u'(c(Z'_{\text{max}}))} \leq \frac{u'(c(Z'_{\text{max}}))}{u'(c(Z'_{\text{max}}))} \leq \frac{u'(c(Z'_{\text{max}}) - (Z'_{\text{max}} - Z'_{\text{min}}))}{u'(c(Z'_{\text{max}}))},
\]

the last inequality since \( c(Z'_{\text{max}}) - c(Z'_{\text{min}}) \leq Z'_{\text{max}} - Z'_{\text{min}} \) and both \( c \) and \( A' \) are increasing in \( Z \). Then

\[
1 \leq \frac{u'(c(Z'_{\text{max}}) - (Z'_{\text{max}} - Z'_{\text{min}}))}{u'(c(Z'_{\text{max}}))} = 1 + \int_0^{Z'_{\text{max}} - Z'_{\text{min}}} \left[ \frac{u''(c(Z'_{\text{max}}) - z)}{u'(c(Z'_{\text{max}}))} \right] dz
\]

\[
= 1 + \int_0^{Z'_{\text{max}} - Z'_{\text{min}}} \left[ \frac{u'(c(Z'_{\text{max}}) - z)}{u'(c(Z'_{\text{max}}))} \cdot \frac{u''(c(Z'_{\text{max}}) - z)}{u'(c(Z'_{\text{max}}) - z))} \right] dz
\]

\[
\to 1
\]

by the assumption \( \lim_{c \to \infty} \frac{u''(c)}{u'(c)} = 0 \).

Q.E.D.
Appendices

Solve

\[
\max_{c,a} u(c) + v(a)
\]
\[
\text{s.t. } c + a \leq W.
\]

Now suppose \(u\) and \(v\) are strictly increasing, concave and differentiable. Attaching the Lagrangian multiplier \(\lambda\) to the constraint, we obtain the f.o.c.’s:

\[
u'(c^*) = v'(a^*) = \lambda,
\]

so \(c^* = f(\lambda)\) and \(a^* = g(\lambda)\) where both \(f\) and \(g\) are decreasing. Plugging this into the budget constraint, we find that \(h(\lambda) = W\), where \(h\) is decreasing. Hence if \(W\) increases, \(\lambda\) decreases, and both \(c^*\) and \(a^*\) increase.

References


